As mentioned earlier, the challenge posed by Hadwiger's conjecture is to devise a proof technique that makes better use of the assumption of $\chi \geqslant r$ than just using its consequence of $\delta \geqslant r-1$ in a suitable subgraph, which we know cannot force a K^r minor (Theorem 7.2.4). So far, no such technique is known.

See correction page 224.

If we resign ourselves to using just $\delta \geqslant r-1$, we can still ask what additional assumptions might help in making this force a K^r minor. Theorem 7.2.7 says that an assumption of large girth has this effect; see also Exercise 32. In fact, a much weaker assumption suffices: for any fixed $s \in \mathbb{N}$ and all large enough d depending only on s, the graphs $G \not\supseteq K_{s,s}$ of average degree at least d can be shown to have K^r minors for r considerably larger than d. For Hadwiger's conjecture, this implies the following:

Theorem 7.3.8. (Kühn & Osthus 2005)
For every integer s there is an integer r_s such that Hadwiger's conjecture holds for all graphs $G \not\supseteq K_{s,s}$ and $r \geqslant r_s$.

The strengthening of Hadwiger's conjecture that graphs of chromatic number at least r contain K^r as a *topological* minor has become known as *Hajós's conjecture*. It is false in general, but Theorem 7.2.7 implies it for graphs of large girth:

Corollary 7.3.9. *There is a constant g such that all graphs G of girth at least g satisfy the implication $\chi(G) \geqslant r \Rightarrow G \supseteq TK^r$ for all r.*

Proof. Let g be the constant from Theorem 7.2.7. If $\chi(G) \geqslant r$ then, by Lemma 5.2.3, G has a subgraph H of minimum degree $\delta(H) \geqslant r-1$. As $g(H) \geqslant g(G) \geqslant g$, Theorem 7.2.7 implies that $G \supseteq H \supseteq TK^r$. $\square$

7.4 Szemerédi's regularity lemma

Some 50 years ago, in the course of the proof of a theorem about arithmetic progressions of integers, Szemerédi developed a graph-theoretical tool that has since come to dominate methods in extremal graph theory like none other: his *regularity lemma*. Very roughly, the lemma says that all graphs can be approximated by random graphs in the following sense: every graph can be partitioned, into a bounded number of equal parts, so that most of its edges run between different parts and the edges between any two parts are distributed fairly uniformly – just as we would expect it if they had been generated at random.

Erdős & Gallai in 1959. It was this result, together with the easy case of stars (Exercise 16) at the other extreme, that inspired the conjecture as a possible unifying result. A proof of the precise conjecture for large graphs was announced in 2009 by Ajtai, Komlós, Simonovits and Szemerédi, but has not been made publicly available.

The Erdős-Sós conjecture says that graphs of average degree greater than $k-1$ contain every tree with k edges. Loebl, Komlós and Sós have conjectured a 'median' version, which appears to be easier: that if at least half the vertices of a graph have degree greater than $k-1$ it contains every tree with k edges. An approximate version of this conjecture has been proved by Hladký, Komlós, Piguet, Simonovits, Stein and Szemerédi in arXiv:1408.3870.

Theorem 7.2.3 was first proved by B. Bollobás & A.G. Thomason, Proof of a conjecture of Mader, Erdős and Hajnal on topological complete subgraphs, *Eur. J. Comb.* **19** (1998), 883–887, and independently by J. Komlós & E. Szemerédi, Topological cliques in graphs II, *Comb. Probab. Comput.* **5** (1996), 79–90. For large G, the latter authors show that the constant c in the theorem can be brought down to about $\frac{1}{2}$, which is not far from the lower bound of $\frac{1}{8}$ given in Exercise 24.

Theorem 7.2.4 was first proved in 1982 by Kostochka, and in 1984 with a better constant by Thomason. For references and more insight, also in these early proofs, see A.G. Thomason, The extremal function for complete minors, *J. Comb. Theory, Ser. B* **81** (2001), 318–338. There, Thomason determines the smallest possible value of the constant c in Theorem 7.2.4 asymptotically for large r. It can be written as $c = \alpha + o(1)$, where $\alpha = 0.53131\ldots$ is an explicit constant and $o(1)$ stands for a function of r tending to zero as $r \to \infty$.

Surprisingly, the average degree needed to force an *in*complete minor H of order r remains at $cr\sqrt{\log r}$, with $c = \alpha\gamma(H) + o(1)$, where γ is a graph invariant $H \mapsto [0,1]$ that is bounded away from 0 for dense H, and $o(1)$ is a function of $|H|$ tending to 0 as $|H| \to \infty$. See J.S. Myers & A.G. Thomason, The extremal function for noncomplete minors, *Combinatorica* **25** (2005), 725–753.

A big step towards the 'linear Hadwiger' conjecture that $\chi \geqslant cr$ forces a K^r minor was obtained by M. Delcourt and L. Postle, Reducing linear Hadwiger's conjecture to coloring small graphs, J. Amer. Math. Soc. (2024), arXiv:2108.01633, who proved that $\chi \geqslant cr \log\log r$ forces a K^r minor.

As Theorem 7.2.4 is best possible, there is no constant c such that all graphs of average degree at least cr have a K^r minor. Strengthening this assumption to $\kappa \geqslant cr$, however, can force a K^r minor in all large enough graphs; this was proved by T. Böhme, K. Kawarabayashi, J. Maharry and B. Mohar, Linear connectivity forces large complete bipartite minors, *J. Comb. Theory, Ser. B* **99** (2009), 557–582. Their proof rests on a structure theorem for graphs of large tree-width not containing a given minor, which was proved only later by R. Diestel, K. Kawarabayashi, Th. Müller & P. Wollan, On the excluded minor structure theorem for graphs of large tree-width, *J. Comb. Theory, Ser. B* **102** (2012), 1189–1210, arXiv:0910.0946. A simple direct argument that bypasses the use of this structure theorem was found by J.-O. Fröhlich and Th. Müller, Linear connectivity forces large complete bipartite minors: an alternative approach, *J. Comb. Theory, Ser. B* **101** (2011), 502–508, arXiv:0906.2568.

The fact that large enough girth can force minors of arbitrarily high minimum degree, and hence large complete minors, was discovered by Thomassen in 1983. The reference can be found in W. Mader, Topological subgraphs in graphs of large girth, *Combinatorica* **18** (1998), 405–412, from which our Lemma 7.2.5 is extracted. Our girth assumption of $8k+3$ has been reduced to more general model-theoretic point of view, and have therefore been based on the strongest of all graph relations, the induced subgraph relation. As a consequence, most of these results are negative; see the notes.

From a graph-theoretic point of view, it seems more promising to look instead for universal graphs for the weaker subgraph relation, or even the topological minor or minor relation. For example, while there is no universal planar graph for subgraphs or induced subgraphs, there is one for minors:

Theorem 8.3.4. (Diestel & Kühn 1999)
There exists a universal planar graph for the minor relation.

This remained the only result about universal graphs for the minor-relation for over 20 years, until Georgakopoulos found $\preccurlyeq$-universal graphs in $\mathrm{Forb}_{\preccurlyeq}(X) = \{\, G \mid X \not\preccurlyeq G \,\}$ ~~when X is K^5, $K_{3,3}$ or $K^{\aleph_0}$~~.

for $X = K^5$ and $X = K_{3,3}$, and proved that there is none for $X = K^{\aleph_0}$.

8.4 Connectivity and matching

In this section we look at infinite versions of Menger's theorem and of the matching theorems from Chapter 2. This area of infinite graph theory is one of its best developed fields, with several deep results. One of these, however, stands out among the rest: a version of Menger's theorem that had been conjectured by Erdős decades ago, and was proved only fairly recently by Aharoni and Berger. The techniques developed for its proof inspired, over the years, much of the theory in this area.

Before we turn to this result, however, let us take a brief look at edge-connectivity. Recall from Section 8.1 that in an infinitely edge-connected countable graph we can easily find infinitely many edge-disjoint spanning trees. Can we still find such trees when the graph is uncountable? We can, but this is not quite as easy to prove (Exercise 62).

The following deep theorem of Laviolette reduces the above problem to its countable case – as it does for many other problems involving edge-connectivity. Let $\mathcal{H}$ be a set of countable graphs forming an edge-decomposition of an arbitrary graph G. Call this decomposition *bond-faithful* if every countable bond of G is contained in some $H \in \mathcal{H}$ and every finite bond of any $H \in \mathcal{H}$ is a bond also of G. Note that the finite bonds of G will be bonds of the $H \in \mathcal{H}$ that contain them. (Why?)

Theorem 8.4.1. (Laviolette 2005)
Every graph has a bond-faithful decomposition into countable graphs.

We shall not be able to prove Laviolette's theorem here. But let us illustrate its power in reducing problems to their countable case by deducing an early classic from the theory of infinite graphs.

Section 12.5. Our new definition seems more natural, since $\leqslant$ implies $\subseteq$ for the sides to which the separations point: if $(A,B) \leqslant (C,D)$ then $B \subseteq C$. This $B \subseteq D$ is also better compatible with the tangle theory of set bipartitions, where it is customary to refer to an oriented partition (A,B) simply as B (since A is determined as $A = V \setminus B$); see the book reference below for more on such tangles and their applications.

Profiles more general than tangles are studied in R. Diestel, F. Hundertmark & S. Lemanczyk, Profiles of separations: in graphs, matroids, and beyond, *Combinatorica* **39,** 37–75. This paper gave the first canonical proof of the tree-of-tangles theorem, Theorem 12.5.1. The tree-of-tangles theorem it proves for profiles of so-called abstract separation systems also implies Theorem 12.3.7 and Exercise 57, since blocks and edge-tangles induce profiles. Indeed this is how they came by their name: as the 'profiles' of blocks visible on the screen of the low-order separations of a graph, which they orient.

Our first proof of the tree-of-tangles theorem, and in particular the splinter lemma on which it is based, are due to C. Elbracht, J. Kneip & M. Teegen, Trees of tangles in abstract separation systems, *J. Comb. Theory, Ser. A* **180** (2021), arXiv:1909.09030. Its canonical strengthening, Theorem 12.5.8, is due to J. Carmesin & J. Kurkofka, Entanglements, *J. Comb. Theory, Ser. B* **164** (2024), 17–28, arXiv:2205.11488. This paper also give examples of entanglements that are not of the form $D(\tau,\tau')$. Thus, Theorem 12.5.8 is also more general than Theorem 12.5.1, not only stronger.

Our proof of Theorem 12.5.9 is adapted from R. Diestel & S. Oum, Tangle-tree duality in abstract separation systems, *Adv. Math.* **377** (2021), 107470; arXiv:1701.02509. In this paper, a duality theory is developed for tangles in abstract separation systems, not necessarily of graphs. Its main result contains Theorems 12.5.9 and 12.5.11 as special cases.

The theory of tangles in graphs, including its main two theorems, has been extended to more general combinatorial structures such as matroids or set partitions. In this general form it can be applied outside mathematics, in areas as diverse as clustering in data analysis, finding mindsets in political science or psychology, or consumer behaviour in economics. This is explored in R. Diestel, *Tangles: a structural approach to artificial intelligence in the empirical sciences*, Cambridge University Press 2024. Excerpts, an electronic edition, and open-source tangle software are available from `tangles-book.com`.

The Kuratowski set for the graphs of tree-width < 4 have been determined by S. Arnborg, D.G. Corneil and A. Proskurowski, Forbidden minors characterization of partial 3-trees, *Discrete Math.* **80** (1990), 1–19. They are: K^5, the octahedron $K_{2,2,2}$, the 5-prism $C^5 \times K^2$, and the Wagner graph W. The Kuratowski set $\mathcal{K}_{\mathcal{P}(S)}$ for a given surface S has been determined explicitly for only one surface other than the sphere, the projective plane. It consists of 35 forbidden minors; see D. Archdeacon, A Kuratowski theorem for the projective plane, *J. Graph Theory* **5** (1981), 243–246. It is not difficult to show that $|\mathcal{K}_{\mathcal{P}(S)}|$ grows rapidly with the genus of S (Exercise 67).

A survey of finite forbidden minor theorems is given in Chapter 6.1 of R. Diestel, *Graph Decompositions*, Oxford University Press 1990. More recent developments are surveyed in R. Thomas, Recent excluded minor theorems, in (J.D. Lamb & D.A. Preece, eds) *Surveys in Combinatorics 1999*, Cambridge University Press 1999, 201–222. A survey of infinite forbidden minor theorems