

## Exercises

- 1.<sup>-</sup> What is the number of edges in a  $K^n$ ?
2. Let  $d \in \mathbb{N}$  and  $V := \{0, 1\}^d$ ; thus,  $V$  is the set of all 0–1 sequences of length  $d$ . The graph on  $V$  in which two such sequences form an edge if and only if they differ in exactly one position is called the  $d$ -dimensional cube. Determine the average degree, number of edges, diameter, girth and circumference of this graph.  
(Hint for the circumference: induction on  $d$ .)
3. Let  $G$  be a graph containing a cycle  $C$ , and assume that  $G$  contains a path of length at least  $k$  between two vertices of  $C$ . Show that  $G$  contains a cycle of length at least  $\sqrt{k}$ .
- 4.<sup>-</sup> Is the bound in Proposition 1.3.2 best possible?
5. Let  $v_0$  be a vertex in a graph  $G$ , and  $D_0 := \{v_0\}$ . For  $n = 1, 2, \dots$  inductively define  $D_n := N_G(D_0 \cup \dots \cup D_{n-1})$ . Show that  $D_n = \{v \mid d(v_0, v) = n\}$  and  $D_{n+1} \subseteq N(D_n) \subseteq D_{n-1} \cup D_{n+1}$  for all  $n \in \mathbb{N}$ .
6. Show that  $\text{rad}(G) \leq \text{diam}(G) \leq 2 \text{rad}(G)$  for every graph  $G$ .
7. Prove the weakening of Theorem 1.3.4 obtained by replacing average with minimum degree. Deduce that  $|G| \geq n_0(d/2, g)$  for every graph  $G$  as given in the theorem.
8. Show that graphs of girth at least 5 and order  $n$  have a minimum degree of  $o(n)$ . In other words, show that there is a function  $f: \mathbb{N} \rightarrow \mathbb{N}$  such that  $f(n)/n \rightarrow 0$  as  $n \rightarrow \infty$  and  $\delta(G) \leq f(n)$  for all such graphs  $G$ .
- 9.<sup>+</sup> Show that every connected graph  $G$  contains a path or cycle of length at least  $\min\{2\delta(G), |G|\}$ .
10. Show that a connected graph of diameter  $k$  and minimum degree  $d$  has at least about  $kd/3$  vertices but need not have substantially more.
- 11.<sup>-</sup> Show that the components of a graph partition its vertex set. (In other words, show that every vertex belongs to exactly one component.)
- 12.<sup>-</sup> Show that every 2-connected graph contains a cycle.
13. Determine  $\kappa(G)$  and  $\lambda(G)$  for  $G = P^m, C^n, K^n, K_{m,n}$  and the  $d$ -dimensional cube (Exercise 2);  $d, m, n \geq 3$ .
- 14.<sup>-</sup> Is there a function  $f: \mathbb{N} \rightarrow \mathbb{N}$  such that, for all  $k \in \mathbb{N}$ , every graph of minimum degree at least  $f(k)$  is  $k$ -connected?
- 15.<sup>+</sup> Let  $\alpha, \beta$  be two graph invariants with positive integer values. Formalize the two statements below, and show that each implies the other:
  - (i)  $\beta$  is bounded above by a function of  $\alpha$ ;
  - (ii)  $\alpha$  can be forced up by making  $\beta$  large enough.

Show that the statement

- (iii)  $\alpha$  is bounded below by a function of  $\beta$

is not equivalent to (i) and (ii). Which small change will make it so?

A. Treglown, Proof of the 1-factorization and Hamilton decomposition conjectures, *Memoirs of the AMS* (to appear).

List colourings were first introduced in 1976 by Vizing. Among other things, Vizing proved the list-colouring equivalent of Brooks's theorem. Voigt (1993) constructed a plane graph of order 238 that is not 4-choosable; thus, Thomassen's list version of the five colour theorem is best possible. A stimulating survey on the list-chromatic number and how it relates to the more classical graph invariants (including a proof of Theorem 5.4.1) is given by N. Alon, Restricted colorings of graphs, in (K. Walker, ed.) *Surveys in Combinatorics*, LMS Lecture Notes **187**, Cambridge University Press 1993. Both the list colouring conjecture and Galvin's proof of the bipartite case are originally stated for multigraphs. Kahn (1994) proved that the conjecture is asymptotically correct, as follows: given any  $\epsilon > 0$ , every graph  $G$  with large enough maximum degree satisfies  $\text{ch}'(G) \leq (1 + \epsilon)\Delta(G)$ .

The total colouring conjecture (Exercise 32) was proposed around 1965 by Vizing and by Behzad; see Jensen & Toft for details.

A gentle introduction to the basic facts about perfect graphs and their applications is given by M.C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press 1980. A more comprehensive treatment is given in A. Schrijver, *Combinatorial optimization*, Springer 2003. Surveys on various aspects of perfect graphs are included in *Perfect Graphs* by J. Ramirez-Alfonsin & B. Reed (eds.), Wiley 2001. Our first proof of the perfect graph theorem, Theorem 5.5.4, follows Lovász's survey on perfect graphs in (L.W. Beineke and R.J. Wilson, eds.) *Selected Topics in Graph Theory 2*, Academic Press 1983. Our second proof, the proof of Theorem 5.5.6, is due to G.S. Gasparian, Minimal imperfect graphs: a simple approach, *Combinatorica* **16** (1996), 209–212.

Theorem 5.5.3 is proved in M. Chudnovsky, N. Robertson, P.D. Seymour and R. Thomas, The strong perfect graph theorem, *Ann. Math.* **164** (2006), 51–229, arXiv:math/0212070. This proof is elucidated by N. Trotignon in his 2013 survey on the arXiv:1301.5149, which also offers a short account of Lovász's proof of the (weak) perfect graph theorem. Chudnovsky, Cornuejols, Liu, Seymour and Vušković, Recognizing Berge graphs, *Combinatorica* **25** (2005), 143–186, constructed an  $O(n^9)$  algorithm testing for odd holes and antiholes, and thus by the strong perfect graph theorem also for perfection.

Gyárfás's conjecture on  $\chi$ -boundedness that prompted Theorem 5.5.7 is from A. Gyárfás, Problems from the world surrounding perfect graphs, Proceedings of the International Conference on Combinatorial Analysis and its Applications (Pokrzywna, 1985), *Zastos. Mat.* **19** (1987), 413–441. Part (i) of the theorem is due to A. Scott and P.D. Seymour, arXiv:1410.4118, while part (ii) was proved by M. Chudnovsky and these authors in arXiv:1506.02232.

The structure of graphs forced by forbidding some fixed induced subgraph or subgraphs, as in the strong perfect graph theorem, has been studied more widely since the proof of that theorem left a community of experts without an overriding goal. One of the central problems now studied is the *Erdős-Hajnal conjecture* that the graphs without some fixed induced subgraph have [linearly large](#) sets of vertices that are either independent or induce a complete subgraph. See Chapter 9.1 for a precise statement.

- (ii)  $V_i \subseteq V_{i-1} \setminus \{v_{i-1}\}$  ( $i = 2, \dots, 2r - 2$ );
- (iii)  $v_{i-1}$  is adjacent either to all vertices in  $V_i$  or to no vertex in  $V_i$  ( $i = 2, \dots, 2r - 2$ ).

Let  $V_1 \subseteq V(G)$  be any set of  $2^{2r-3}$  vertices, and pick  $v_1 \in V_1$  arbitrarily. Then (i) holds for  $i = 1$ , while (ii) and (iii) hold trivially. Suppose now that  $V_{i-1}$  and  $v_{i-1} \in V_{i-1}$  have been chosen so as to satisfy (i)–(iii) for  $i - 1$ , where  $1 < i \leq 2r - 2$ . Since

$$|V_{i-1} \setminus \{v_{i-1}\}| = 2^{2r-1-i} - 1$$

is odd,  $V_{i-1}$  has a subset  $V_i$  satisfying (i)–(iii); we pick  $v_i \in V_i$  arbitrarily.

Among the  $2r - 3$  vertices  $v_1, \dots, v_{2r-3}$ , there are  $r - 1$  vertices that show the same behaviour when viewed as  $v_{i-1}$  in (iii), being adjacent either to all the vertices in  $V_i$  or to none. Accordingly, these  $r - 1$  vertices and  $v_{2r-2}$  induce either a  $K^r$  or a  $\overline{K}^r$  in  $G$ , because  $v_i, \dots, v_{2r-2} \in V_i$  for all  $i$ .  $\square$

The least integer  $n$  associated with  $r$  as in Theorem 9.1.1 is the *Ramsey number*  $R(r)$  of  $r$ ; our proof shows that  $R(r) \leq 2^{2r-3}$ . In Chapter 11 we shall use a simple probabilistic argument to show that  $R(r)$  is bounded below by  $2^{r/2}$  (Theorem 11.1.3).

Ramsey  
number  
 $R(r)$

In other words, the largest clique or independent set of vertices that a graph of order  $n$  must contain is, asymptotically, logarithmically small in  $n$ . As soon as we forbid some fixed induced subgraph, however, it may be much bigger, ~~of size linear in  $n$~~ : The *Erdős-Hajnal conjecture* says that for every graph  $H$  there exists a constant  $\delta_H > 0$  such that every graph  $G$  not containing an induced copy of  $H$  has a set of at least  $|G|^{\delta_H}$  vertices that are either independent or span a complete subgraph in  $G$ .

Erdős-  
Hajnal  
conjecture

It is customary in Ramsey theory to think of partitions as colourings: a *colouring* of (the elements of) a set  $X$  with  $c$  colours, or  $c$ -colouring for short, is simply a partition of  $X$  into  $c$  classes (indexed by the ‘colours’). In particular, these colourings need not satisfy any non-adjacency requirements as in Chapter 5. Given a  $c$ -colouring of  $[X]^k$ , the set of all  $k$ -subsets of  $X$ , we call a set  $Y \subseteq [X]^k$  *monochromatic* if all the elements of  $Y$  have the same colour,<sup>1</sup> i.e. belong to the same of the  $c$  partition classes of  $[X]^k$ . Similarly, if  $G = (V, E)$  is a graph and all the edges of  $H \subseteq G$  have the same colour in some colouring of  $E$ , we call  $H$  a *monochromatic subgraph* of  $G$ , speak of a red (green, etc.)  $H$  in  $G$ , and so on.

$c$ -colouring

$[X]^k$

mono-  
chromatic

In the above terminology, Ramsey's theorem can be expressed as follows: for every  $r$  there exists an  $n$  such that, given any  $n$ -set  $X$ ,

<sup>1</sup> Note that  $Y$  is called monochromatic, but it is the elements of  $[Y]^k$ , not of  $Y$ , that are (equally) coloured.