1.7 Contraction and minors

In Section 1.1 we saw two fundamental containment relations between graphs: the 'subgraph' relation, and the 'induced subgraph' relation. In this section we meet two more: the 'minor' relation, and the 'topological minor' relation. Let X be a fixed graph.

subdivision TX *of* X

A *subdivision* of X is, informally, any graph obtained from X by 'subdividing' some or all of its edges by drawing new vertices on those edges. In other words, we replace some edges of X with new paths between their ends, so that none of these paths has an inner vertex in $V(X)$ or on another new path. When G is a subdivision of X, we also say that G *is a* TX.[7] The original vertices of X are the *branch vertices of the* TX; its new vertices are called *subdividing vertices*. Note that subdividing vertices have degree 2, while branch vertices retain their degree from X.

branch vertices

topological minor

If a graph Y contains a TX as a subgraph, then X is a *topological minor* of Y (Fig. 1.7.1).

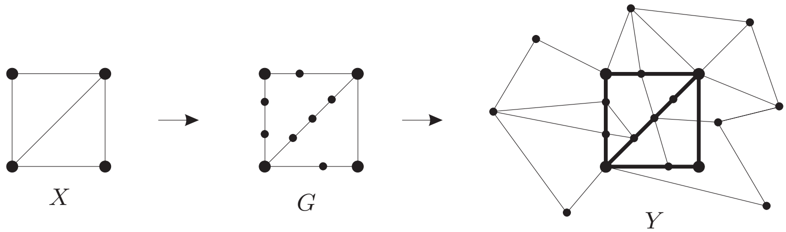

Fig. 1.7.1. The graph G is a TX, a *subdivision* of X.
As $G \subseteq Y$, this makes X a *topological minor* of Y.

Similarly, replacing the vertices x of X with disjoint connected graphs G_x, and the edges xy of X with non-empty sets of G_x–G_y edges, yields a graph that we shall call an IX.[8] More formally, a graph G *is an* IX if its vertex set admits a partition $\{ V_x \mid x \in V(X) \}$ into connected subsets V_x such that distinct vertices $x, y \in X$ are adjacent in X if and only if G contains a V_x–V_y edge. The sets V_x are the *branch sets of the* IX. Conversely, we say that X arises from G by *contracting* the subgraphs G_x and call it a *contraction minor* of $\cancel{Y}$.

IX

branch sets

contraction

If a graph Y contains an IX as a subgraph, then X is a *minor* of Y, the IX is a *model of* X in Y, and we write $X \preccurlyeq Y$ (Fig. 1.7.2).

minor, $\preccurlyeq$ *model*

[7] The 'T' stands for 'topological'. Although, formally, TX denotes a whole class of graphs, the class of all subdivisions of X, it is customary to use the expression as indicated to refer to an arbitrary member of that class.

[8] The 'I' stands for 'inflated'. As before, while IX is formally a class of graphs, those admitting a vertex partition $\{ V_x \mid x \in V(X) \}$ as described below, we use the expression as indicated to refer to an arbitrary member of that class.

Exercises

1.$^{-}$ What is the number of edges in a K^n?

2. Let $d \in \mathbb{N}$ and $V := \{0,1\}^d$; thus, V is the set of all 0–1 sequences of length d. The graph on V in which two such sequences form an edge if and only if they differ in exactly one position is called the *d-dimensional cube*. Determine the average degree, number of edges, diameter, girth and circumference of this graph.

   (Hint for the circumference: induction on d.)

3. Let G be a graph containing a cycle C, and assume that G contains a path of length at least k between two vertices of C. Show that G contains a cycle of length at least $\sqrt{k}$.

4.$^{-}$ Is the bound in Proposition 1.3.2 best possible?

5. Let v_0 be a vertex in a graph G, and $D_0 := \{v_0\}$. For $n = 1, 2, \ldots$ inductively define $D_n := N_G(D_0 \cup \ldots \cup D_{n-1})$. Show that $D_n = \{ v \mid d(v_0, v) = n \}$ and $D_{n+1} \subseteq N(D_n) \subseteq D_{n-1} \cup D_{n+1}$ for all $n \in \mathbb{N}$.

6. Show that $\mathrm{rad}(G) \leqslant \mathrm{diam}(G) \leqslant 2\,\mathrm{rad}(G)$ for every graph G.

7. Prove the weakening of Theorem 1.3.4 obtained by replacing average with minimum degree. Deduce that $|G| \geqslant n_0(d/2, g)$ for every graph G as given in the theorem.

8. Show that graphs of girth at least 5 and order n have a minimum degree of $o(n)$. In other words, show that there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(n)/n \to 0$ as $n \to \infty$ and $\delta(G) \leqslant f(n)$ for all such graphs G.

9.$^{+}$ Show that every connected graph G contains a path or cycle of length at least $\min\{2\delta(G), |G|\}$.

10. Show that a connected graph of diameter k and minimum degree d has at least about $kd/3$ vertices but need not have substantially more.

11.$^{-}$ Show that the components of a graph partition its vertex set. (In other words, show that every vertex belongs to exactly one component.)

12.$^{-}$ Show that every 2-connected graph contains a cycle.

13. Determine $\kappa(G)$ and $\lambda(G)$ for $G = P^m, C^n, K^n, K_{m,n}$ and the d-dimensional cube (Exercise 2); $d, m, n \geqslant 3$.

14.$^{-}$ Is there a function $f\colon \mathbb{N} \to \mathbb{N}$ such that, for all $k \in \mathbb{N}$, every graph of minimum degree at least $f(k)$ is k-connected?

15.$^{+}$ Let α, β be two graph invariants with positive integer values. Formalize the two statements below, and show that each implies the other:

    (i) β is bounded above by a function of α;

    (ii) α can be forced up by making β large enough.

    Show that the statement

    (iii) α is bounded below by a function of β

    is not equivalent to (i) and (ii). Which small change will make it so?

A. Treglown, Proof of the 1-factorization and Hamilton decomposition conjectures, *Memoirs of the AMS* (to appear).

List colourings were first introduced in 1976 by Vizing. Among other things, Vizing proved the list-colouring equivalent of Brooks's theorem. Voigt (1993) constructed a plane graph of order 238 that is not 4-choosable; thus, Thomassen's list version of the five colour theorem is best possible. A stimulating survey on the list-chromatic number and how it relates to the more classical graph invariants (including a proof of Theorem 5.4.1) is given by N. Alon, Restricted colorings of graphs, in (K. Walker, ed.) *Surveys in Combinatorics*, LMS Lecture Notes **187**, Cambridge University Press 1993. Both the list colouring conjecture and Galvin's proof of the bipartite case are originally stated for multigraphs. Kahn (1994) proved that the conjecture is asymptotically correct, as follows: given any $\epsilon > 0$, every graph G with large enough maximum degree satisfies $\mathrm{ch}'(G) \leqslant (1+\epsilon)\Delta(G)$.

The total colouring conjecture (Exercise **32**) was proposed around 1965 by Vizing and by Behzad; see Jensen & Toft for details.

A gentle introduction to the basic facts about perfect graphs and their applications is given by M.C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press 1980. A more comprehensive treatment is given in A. Schrijver, *Combinatorial optimization*, Springer 2003. Surveys on various aspects of perfect graphs are included in *Perfect Graphs* by J. Ramirez-Alfonsin & B. Reed (eds.), Wiley 2001. Our first proof of the perfect graph theorem, Theorem 5.5.4, follows Lovász's survey on perfect graphs in (L.W. Beineke and R.J. Wilson, eds.) *Selected Topics in Graph Theory 2*, Academic Press 1983. Our second proof, the proof of Theorem 5.5.6, is due to G.S. Gasparian, Minimal imperfect graphs: a simple approach, *Combinatorica* **16** (1996), 209–212.

Theorem 5.5.3 is proved in M. Chudnovsky, N. Robertson, P.D. Seymour and R. Thomas, The strong perfect graph theorem, *Ann. Math.* **164** (2006), 51–229, arXiv:math/0212070. This proof is elucidated by N. Trotignon in his 2013 survey on the arXiv:1301.5149, which also offers a short account of Lovász's proof of the (weak) perfect graph theorem. Chudnovsky, Cornuejols, Liu, Seymour and Vušković, Recognizing Berge graphs, *Combinatorica* **25** (2005), 143–186, constructed an $O(n^9)$ algorithm testing for odd holes and antiholes, and thus by the strong perfect graph theorem also for perfection.

Gýarfás's conjecture on χ-boundedness that prompted Theorem 5.5.7 is from A. Gyárfás, Problems from the world surrounding perfect graphs, Proceedings of the International Conference on Combinatorial Analysis and its Applications (Pokrzywna, 1985), *Zastos. Mat.* **19** (1987), 413–441. Part (i) of the theorem is due to A. Scott and P.D. Seymour, arXiv:1410.4118, while part (ii) was proved by M. Chudnovsky and these authors in arXiv:1506.02232.

The structure of graphs forced by forbidding some fixed induced subgraph or subgraphs, as in the strong perfect graph theorem, has been studied more widely since the proof of that theorem left a community of experts without an overriding goal. One of the central problems now studied is the *Erdős-Hajnal conjecture* that the graphs without some fixed induced subgraph ~~have linearly large~~ sets of vertices that are either independent or induce a complete subgraph. See Chapter 9.1 for a precise statement.

(ii) $V_i \subseteq V_{i-1} \smallsetminus \{v_{i-1}\}$ $(i = 2, \dots, 2r-2)$;

(iii) v_{i-1} is adjacent either to all vertices in V_i or to no vertex in V_i $(i = 2, \dots, 2r-2)$.

Let $V_1 \subseteq V(G)$ be any set of 2^{2r-3} vertices, and pick $v_1 \in V_1$ arbitrarily. Then (i) holds for $i = 1$, while (ii) and (iii) hold trivially. Suppose now that V_{i-1} and $v_{i-1} \in V_{i-1}$ have been chosen so as to satisfy (i)–(iii) for $i-1$, where $1 < i \leqslant 2r-2$. Since

$$|V_{i-1} \smallsetminus \{v_{i-1}\}| = 2^{2r-1-i} - 1$$

is odd, V_{i-1} has a subset V_i satisfying (i)–(iii); we pick $v_i \in V_i$ arbitrarily.

Among the $2r-3$ vertices $v_1, \dots, v_{2r-3}$, there are $r-1$ vertices that show the same behaviour when viewed as v_{i-1} in (iii), being adjacent either to all the vertices in V_i or to none. Accordingly, these $r-1$ vertices and v_{2r-2} induce either a K^r or a $\overline{K^r}$ in G, because $v_i, \dots, v_{2r-2} \in V_i$ for all i. □

The least integer n associated with r as in Theorem 9.1.1 is the *Ramsey number* $R(r)$ of r; our proof shows that $R(r) \leqslant 2^{2r-3}$. In Chapter 11 we shall use a simple probabilistic argument to show that $R(r)$ is bounded below by $2^{r/2}$ (Theorem 11.1.3). *Ramsey number* $R(r)$

In other words, the largest clique or independent set of vertices that a graph of order n must contain is, asymptotically, logarithmically small in n. As soon as we forbid some fixed induced subgraph, however, it may be much bigger, ~~of size linear in n~~: The *Erdős-Hajnal conjecture* says that for every graph H there exists a constant $\delta_H > 0$ such that every graph G not containing an induced copy of H has a set of at least $|G|^{\delta_H}$ vertices that are either independent or span a complete subgraph in G. *Erdős-Hajnal conjecture*

It is customary in Ramsey theory to think of partitions as colourings: a *colouring* of (the elements of) a set X *with c colours*, or *c-colouring* for short, is simply a partition of X into c classes (indexed by the 'colours'). In particular, these colourings need not satisfy any non-adjacency requirements as in Chapter 5. Given a c-colouring of $[X]^k$, the set of all k-subsets of X, we call a set $Y \subseteq X$ *monochromatic* if all the elements of $[Y]^k$ have the same colour,[1] i.e. belong to the same of the c partition classes of $[X]^k$. Similarly, if $G = (V, E)$ is a graph and all the edges of $H \subseteq G$ have the same colour in some colouring of E, we call H a *monochromatic subgraph* of G, speak of a red (green, etc.) H in G, and so on. *c-colouring* $[X]^k$ *monochromatic*

In the above terminology, Ramsey's theorem can be expressed as follows: for every r there exists an n such that, given any n-set X,

[1] Note that Y is called monochromatic, but it is the elements of $[Y]^k$, not of Y, that are (equally) coloured.