Lemma 1.5.5. Let $T$ be a normal tree in $G$.

(i) Any two vertices $x, y \in T$ are separated in $G$ by the set $[x] \cap [y]$.

(ii) If $S \subseteq V(T) = V(G)$ and $S$ is down-closed, then the components of $G - S$ are spanned by the sets $[x]$ with $x$ minimal in $T - S$.

Proof. (i) Let $P$ be any $x$–$y$ path in $G$; we show that $P$ meets $[x] \cap [y]$.

Let $t_1, \ldots, t_n$ be a minimal sequence of vertices in $P \cap T$ such that $t_1 = x$ and $t_n = y$ and $t_i$ and $t_{i+1}$ are comparable in the tree-order of $T$ for all $i$. (Such a sequence exists: the set of all vertices in $P \cap T$, in their natural order as they occur on $P$, has this property because $T$ is normal and every segment $t_iP_{i+1}$ is either an edge of $T$ or a $T$-path.) In our minimal sequence we cannot have $t_{i-1} < t_i > t_{i+1}$ for any $i$, since $t_{i-1}$ and $t_{i+1}$ would then be comparable, and deleting $t_i$ would yield a smaller such sequence. Thus, our sequence has the form

$$x = t_1 \ldots > t_k < \ldots < t_n = y$$

for some $k \in \{1, \ldots, n\}$. As $t_k \in [x] \cap [y] \cap V(P)$, our proof is complete.

(ii) Consider a component $C$ of $G - S$, and let $x$ be a minimal element of its vertex set. Then $V(C)$ has no other minimal element $x'$: as $x$ and $x'$ would be incomparable, any $x$–$x'$ path in $C$ would by (i) contain a vertex below both, contradicting their minimality in $V(C)$. Hence as every vertex of $C$ lies above some minimal element of $V(C)$, it lies above $x$. Conversely, every vertex $y \in [x]$ lies in $C$, for since $S$ is down-closed, the ascending path $xTy$ lies in $T - S$. Thus, $V(C) = [x]$.

Let us show that $x$ is minimal not only in $V(C)$ but also in $T - S$. The vertices below $x$ form a chain $[t]$ in $T$. As $t$ is a neighbour of $x$, the maximality of $C$ as a component of $G - S$ implies that $t \in S$, giving $[t] \subseteq S$ since $S$ is down-closed. This completes the proof that every component of $G - S$ is spanned by a set $[x]$ with $x$ minimal in $T - S$.

Conversely, if $x$ is any minimal element of $T - S$, it is clearly also minimal in the component $C$ of $G - S$ to which it belongs. Then $V(C) = [x]$ as before, i.e., $[x]$ spans this component. $\square$

Normal spanning trees are also called depth-first search trees, because of the way they arise in computer searches on graphs (Exercise 26). This fact is often used to prove their existence, which can also be shown by a very short and clever induction (Exercise 25). The following constructive proof, however, illuminates better how normal trees capture the structure of their host graphs.

Proposition 1.5.6. Every connected graph contains a normal spanning tree, with any specified vertex as its root.
1.7 Contraction and minors

In Section 1.1 we saw two fundamental containment relations between graphs: the ‘subgraph’ relation, and the ‘induced subgraph’ relation. In this section we meet two more: the ‘minor’ relation, and the ‘topological minor’ relation. Let \( X \) be a fixed graph.

A subdivision of \( X \) is, informally, any graph obtained from \( X \) by ‘subdividing’ some or all of its edges by drawing new vertices on those edges. In other words, we replace some edges of \( X \) with new paths subdivision \( TX \) of \( X \) between their ends, so that none of these paths has an inner vertex in \( V(X) \) or on another new path. When \( G \) is a subdivision of \( X \), we also say that \( G \) is a \( TX \). The original vertices of \( X \) are the branch vertices branch \( TX \); its new vertices are called subdividing vertices. Note that subdividing vertices have degree 2, while branch vertices retain their degree from \( X \).

If a graph \( Y \) contains a \( TX \) as a subgraph, then \( X \) is a topological minor of \( Y \) (Fig. 1.7.1).

Similarly, replacing the vertices \( x \) of \( X \) with disjoint connected graphs \( G_x \), and the edges \( xy \) of \( X \) with non-empty sets of \( G_x-G_y \) edges, yields a graph that we shall call an \( IX \). More formally, a graph \( G \) is an \( IX \) if its vertex set admits a partition \( \{ V_x \mid x \in V(X) \} \) into connected subsets \( V_x \) such that distinct vertices \( x, y \in X \) are adjacent in \( X \) if and only if \( G \) contains a \( V_x-V_y \) edge. The sets \( V_x \) are the branch sets branch of the \( IX \). Conversely, we say that \( X \) arises from \( G \) by contracting the subgraphs \( G_x \) and call it a contraction minor of \( Y \).

If a graph \( Y \) contains an \( IX \) as a subgraph, then \( X \) is a minor of \( Y \), the \( IX \) is a model of \( X \) in \( Y \), and we write \( X \preceq Y \) (Fig. 1.7.2).

---

7 The ‘\( T \)’ stands for ‘topological’. Although, formally, \( TX \) denotes a whole class of graphs, the class of all subdivisions of \( X \), it is customary to use the expression as indicated to refer to an arbitrary member of that class.

8 The ‘\( I \)’ stands for ‘inflated’. As before, while \( IX \) is formally a class of graphs, those admitting a vertex partition \( \{ V_x \mid x \in V(X) \} \) as described below, we use the expression as indicated to refer to an arbitrary member of that class.
Exercises

1. What is the number of edges in a $K^n$?

2. Let $d \in \mathbb{N}$ and $V := \{0, 1\}^d$; thus, $V$ is the set of all 0–1 sequences of length $d$. The graph on $V$ in which two such sequences form an edge if and only if they differ in exactly one position is called the $d$-dimensional cube. Determine the average degree, number of edges, diameter, girth and circumference of this graph.

(Hint for the circumference: induction on $d$.)

3. Let $G$ be a graph containing a cycle $C$, and assume that $G$ contains a path of length at least $k$ between two vertices of $C$. Show that $G$ contains a cycle of length at least $\sqrt{k}$.

4. Is the bound in Proposition 1.3.2 best possible?

5. Let $v_0$ be a vertex in a graph $G$, and $D_0 := \{v_0\}$. For $n = 1, 2, \ldots$ inductively define $D_n := N_G(D_0 \cup \ldots \cup D_{n-1})$. Show that $D_n = \{v \mid d(v_0, v) = n\}$ and $D_{n+1} \subseteq N(D_n) \subseteq D_{n-1} \cup D_{n+1}$ for all $n \in \mathbb{N}$.

6. Show that $\text{rad}(G) \leq \text{diam}(G) \leq 2 \text{rad}(G)$ for every graph $G$.

7. Prove the weakening of Theorem 1.3.4 obtained by replacing average with minimum degree. Deduce that $|G| \geq n_0(d/2, g)$ for every graph $G$ as given in the theorem.

8. Show that graphs of girth at least 5 and order $n$ have a minimum degree of $o(n)$. In other words, show that there is a function $f: \mathbb{N} \to \mathbb{N}$ such that $f(n)/n \to 0$ as $n \to \infty$ and $\delta(G) \leq f(n)$ for all such graphs $G$.

9. Show that every connected graph $G$ contains a path or cycle of length at least $\min\{2\delta(G), |G|\}$.

10. Show that a connected graph of diameter $k$ and minimum degree $d$ has at least about $kd/3$ vertices but need not have substantially more.

11. Show that the components of a graph partition its vertex set. (In other words, show that every vertex belongs to exactly one component.)

12. Show that every 2-connected graph contains a cycle.

13. Determine $\kappa(G)$ and $\lambda(G)$ for $G = P^n, C^n, K^n, K_{m,n}$ and the $d$-dimensional cube (Exercise 2); $d, m, n \geq 3$.

14. Is there a function $f: \mathbb{N} \to \mathbb{N}$ such that, for all $k \in \mathbb{N}$, every graph of minimum degree at least $f(k)$ is $k$-connected?

15. Let $\alpha, \beta$ be two graph invariants with positive integer values. Formalize the two statements below, and show that each implies the other:

   (i) $\beta$ is bounded above by a function of $\alpha$;

   (ii) $\alpha$ can be forced up by making $\beta$ large enough.

   Show that the statement

   (iii) $\alpha$ is bounded below by a function of $\beta$

   is not equivalent to (i) and (ii). Which small change will make it so?

List colourings were first introduced in 1976 by Vizing. Among other things, Vizing proved the list-colouring equivalent of Brooks’s theorem. Voigt (1993) constructed a planar graph of order 238 that is not 4-choosable; thus, Thomassen’s list version of the five colour theorem is best possible. A stimulating survey on the list-chromatic number and how it relates to the more classical graph invariants (including a proof of Theorem 5.4.1) is given by N. Alon, Restricted colorings of graphs, in (K. Walker, ed.) *Surveys in Combinatorics*, LMS Lecture Notes 187, Cambridge University Press 1993. Both the list colouring conjecture and Galvin’s proof of the bipartite case are originally stated for multigraphs. Kahn (1994) proved that the conjecture is asymptotically correct, as follows: given any $\epsilon > 0$, every graph $G$ with large enough maximum degree satisfies $\chi'(G) \leq (1 + \epsilon)\Delta(G)$.

The total colouring conjecture (Exercise 32) was proposed around 1965 by Vizing and by Behzad; see Jensen & Toft for details.


Gyárfás’s conjecture on $\chi$-boundedness that prompted Theorem 5.5.7 is from A. Gyárfás, Problems from the world surrounding perfect graphs, Proceedings of the International Conference on Combinatorial Analysis and its Applications (Pokrzywna, 1985), *Zastos. Mat.* 19 (1987), 413–441. Part (i) of the theorem is due to A. Scott and P.D. Seymour, arXiv:1410.4118, while part (ii) was proved by M. Chudnovsky and these authors in arXiv:1506.02232.

The structure of graphs forced by forbidding some fixed induced subgraph or subgraphs, as in the strong perfect graph theorem, has been studied more widely since the proof of that theorem left a community of experts without an overriding goal. One of the central problems now studied is the *Erdős-Hajnal conjecture* that the graphs without some fixed induced subgraph have *linearly large* sets of vertices that are either independent or induce a complete subgraph. See Chapter 9.1 for a precise statement.
vertices outside \( U \cup W \) that are not adjacent to \( u \), and let \( V_2 \) contain the remaining vertices. As \( u \) is isolated in \( G[V_1] \), we have \( G \not\simeq G[V_1] \) and hence \( G \simeq G[V_2] \). By the minimality of \( |U \cup W| \), there is a vertex \( v \in G[V_2] - U - W \) that is adjacent to every vertex in \( U \setminus \{u\} \) and to none in \( W \). But \( v \) is also adjacent to \( u \), because it lies in \( V_2 \). So \( U, W, v \) satisfy (*) for \( G \), contrary to assumption.

Finally, assume that \( U = \emptyset \). Then \( W \neq \emptyset \). Pick \( w \in W \), and consider the partition \( \{V_1, V_2\} \) of \( V \) where \( V_1 \) consists of \( w \) and all its neighbours outside \( W \). As before, \( G \not\simeq G[V_1] \) and hence \( G \simeq G[V_2] \). Therefore \( U \) and \( W \setminus \{w\} \) satisfy (*) in \( G[V_2] \), with \( v \in V_2 \setminus W \) say, and then \( U, W, v \) satisfy (*) in \( G \). \( \square \)

Another indication of the high degree of uniformity in the structure of the Rado graph is its large automorphism group. For example, \( R \) is easily seen to be vertex-transitive: given any two vertices \( x \) and \( y \), there is an automorphism of \( R \) mapping \( x \) to \( y \).

In fact, much more is true: using the back-and-forth technique, one can easily show that the Rado graph is homogeneous: every isomorphism between two finite induced subgraphs can be extended to an automorphism of the entire graph (Exercise 50).

Which other countable graphs are homogeneous? The complete graph \( K_{\aleph_0} \) and its complement are again obvious examples. Moreover, for every integer \( r \geq 3 \) there is a homogeneous \( K^r \)-free graph \( R^r \), constructed as follows. Let \( R^r_0 := K^1 \), and let \( R^r_{n+1} \) be obtained from \( R^r_n \) by joining, for every subgraph \( H \nsubseteq K^{r-1} \) of \( R^r_n \), a new vertex \( v_H \) to every vertex in \( H \). Then let \( R^r := \bigcup_{n \in \mathbb{N}} R^r_n \). Clearly, as the new vertices \( v_H \) of \( R^r_{n+1} \) are independent, there is no \( K^r \) in \( R^r_{n+1} \) if there was none in \( R^r_n \), so \( R^r \nsubseteq K^r \) by induction on \( n \). Just like the Rado graph, \( R^r \) is clearly universal among the \( K^r \)-free countable graphs, and by the back-and-forth argument from the proof of Theorem 8.3.1 it is easily seen to be homogeneous.

By the following deep theorem of Lachlan and Woodrow, the countable homogeneous graphs we have seen so far are essentially all:

**Theorem 8.3.3.** (Lachlan & Woodrow 1980)
Every countably infinite homogeneous graph is one of the following:

- a disjoint union of complete graphs of the same order, or the complement of such a graph;
- the graph \( R^r \) or its complement, for some \( r \geq 3 \);
- the Rado graph \( R \).

To conclude this section, let us return to our original problem: for which graph properties is there a graph that is universal with this property? Most investigations into this problem have addressed it from a
Theorem 8.5.3. Every countable rayless graph \( G \) has an unfriendly partition.

Proof. To help with our formal notation, we shall think of a partition of a set \( V \) as a map \( \pi : V \to \{0, 1\} \). We apply induction on the rank of \( G \). When this is zero then \( G \) is finite, and an unfriendly partition can be obtained by maximizing the number of edges across the partition.

Suppose now that \( G \) has rank \( \alpha > 0 \), and assume the theorem as true for graphs of smaller rank.

Let \( U \) be a finite set of vertices in \( G \) such that each of the components \( C_0, C_1, \ldots \) of \( G - U \) has rank \( < \alpha \). Partition \( U \) into the set \( U_0 \) of vertices that have finite degree in \( G \), the set \( U_1 \) of vertices that have infinitely many neighbours in some \( C_n \), and the set \( U_2 \) of vertices that have infinite degree but only finitely many neighbours in each \( C_n \).

\( G_0, G_1, \ldots \) For every \( n \in \mathbb{N} \) let \( G_n := G[U \cup V(C_n) \cup \ldots \cup V(C_n)] \). This is a graph of some rank \( \alpha_n < \alpha \), so by induction it has an unfriendly partition \( \pi_n \). Each of these \( \pi_n \) induces a partition of \( U \). Let \( \pi_U \) be a partition of \( U \) induced by \( \pi_n \) for infinitely many \( n \), say for \( n_0 < n_1 < \ldots \). Choose \( n_0 \) large enough that \( G_{n_0} \) contains all the neighbours of vertices in \( U_0 \), and the other \( n_1 \) large enough that every vertex in \( U_2 \) has more neighbours in \( G_{n_i} - G_{n_{i-1}} \) than in \( G_{n_{i-1}} \), for all \( i > 0 \). Let \( \pi \) be the partition of \( G \) defined by letting \( \pi(v) := \pi_{n_i}(v) \) for all \( v \in G_{n_i} - G_{n_{i-1}} \) and all \( i \), where \( G_{n_{-1}} := \emptyset \). Note that \( \pi|_U = \pi_{n_0}|_U = \pi_U \).

Let us show that \( \pi \) is unfriendly. We have to check that every vertex is happy with \( \pi \), i.e., that it has at least as many neighbours in the opposite class under \( \pi \) as in its own.\(^8\) To see that a vertex \( v \in G - U \) is happy with \( \pi \), let \( i \) be minimal such that \( v \in G_{n_i} \), and recall that \( v \) was happy with \( \pi_{n_i} \). As both \( v \) and its neighbours in \( G \) lie in \( U \cup V(G_{n_i} - G_{n_{i-1}}) \), and \( \pi \) agrees with \( \pi_{n_i} \) on this set, \( v \) is happy also with \( \pi \). Vertices in \( U_0 \) are happy with \( \pi \), because they were happy with \( \pi_{n_0} \), and \( \pi \) agrees with \( \pi_{n_0} \) on \( U_0 \) and all its neighbours. Vertices in \( U_1 \) are also happy. Indeed, every \( u \in U_1 \) has infinitely many neighbours in some \( C_n \), and hence in some \( G_{n_i} - G_{n_{i-1}} \). Then \( u \) has infinitely many opposite neighbours in \( G_{n_i} - G_{n_{i-1}} \) under \( \pi_{n_i} \). Since \( \pi_{n_i} \) agrees with \( \pi \) on both \( U \) and \( G_{n_i} - G_{n_{i-1}} \), our vertex \( u \) has infinitely many opposite neighbours also under \( \pi \). Vertices in \( U_2 \), finally, are happy with every \( \pi_{n_i} \). By our choice of \( n_i \), at least one of their opposite neighbours under \( \pi_{n_i} \) must lie in \( G_{n_i} - G_{n_{i-1}} \). Since \( \pi_{n_i} \) agrees with \( \pi \) on both \( U_2 \) and \( G_{n_i} - G_{n_{i-1}} \), this gives every \( u \in U_2 \) at least one opposite neighbour under \( \pi \) in every \( G_{n_i} - G_{n_{i-1}} \). Hence \( u \) has infinitely many opposite neighbours under \( \pi \), which clearly makes it happy.

\(^8\) It is only by tradition that such partitions are called ‘unfriendly’; our vertices love them.
30. Show that a graph $G$ contains a $TK^0_0$ if and only if some end of $G$ is dominated by infinitely many vertices.

31. $+$ Let $G$ be a finitely separable graph, one in which any two vertices can be separated by finitely many edges.

(i) Show that any two rays in $G$ that cannot be separated by finitely many edges are dominated by a common vertex.

(ii) Is the assumption of finite separability necessary for (i) to hold?

32. Construct a countable graph with uncountably many thick ends. Can you find a locally finite such graph?

33. Show that a locally finite connected vertex-transitive graph has exactly 0, 1, 2 or infinitely many ends.

34. $+$ Show that the automorphisms of a graph $G = (V, E)$ act naturally on its ends, i.e., that every automorphism $\sigma : V \to V$ can be extended to a map $\sigma : \Omega(G) \to \Omega(G)$ such that $\sigma(R) \in \sigma(\omega)$ whenever $R$ is a ray in an end $\omega$. Prove that, if $G$ is connected, every automorphism $\sigma$ of $G$ fixes a finite set of vertices or an end. If $\sigma$ fixes no finite set of vertices, can it fix more than one end? More than two?

35. $-$ Show that a locally finite spanning tree of a graph $G$ contains a ray from every end of $G$.

36. A ray in a graph follows another ray if the two have infinitely many vertices in common. Show that if $T$ is a normal spanning tree of $G$ then every ray of $G$ follows a unique normal ray of $T$.

37. Use normal spanning trees to show that a countable connected graph has either countably many or continuum many ends.

38. Show that the following assertions are equivalent for connected countable graphs $G$.

(i) $G$ has a locally finite spanning tree.

(ii) For no finite separator $X \subseteq V(G)$ in $G$ does $G - X$ have infinitely many components.

39. Show that every (countable) planar 3-connected graph has a locally finite spanning tree.

40. Prove the following infinite version of the Erdős-Pósa theorem: an infinite graph $G$ either contains infinitely many disjoint cycles or it has a finite set $Z$ of vertices such that $G - Z$ is a forest.

41. Let $G$ be a connected graph. Call a set $U \subseteq V(G)$ dispersed if every ray in $G$ can be separated from $U$ by a finite set of vertices. (In the topology of Section 8.6, these are precisely the closed subsets of $V(G)$.)

(i) Prove that $G$ has a normal spanning tree if and only if $V(G)$ is a countable union of dispersed sets.

(ii) Deduce that if $G$ has a normal spanning tree then so does every connected minor of $G$. 
Let \( V_i \subseteq V(G) \) be any set of \( 2^{2r-3} \) vertices, and pick \( v_1 \in V_1 \) arbitrarily. Then (i) holds for \( i = 1 \), while (ii) and (iii) hold trivially. Suppose now that \( V_{i-1} \) and \( v_{i-1} \in V_{i-1} \) have been chosen so as to satisfy (i)–(iii) for \( i-1 \), where \( 1 < i \leq 2r - 2 \). Since

\[
|V_{i-1} \setminus \{ v_{i-1} \}| = 2^{2r-1-i-1}
\]

is odd, \( V_{i-1} \) has a subset \( V_i \) satisfying (i)–(iii); we pick \( v_i \in V_i \) arbitrarily.

Among the \( 2^{r-3} \) vertices \( v_1, \ldots, v_{2r-3} \), there are \( r-1 \) vertices that show the same behaviour when viewed as \( v_{i-1} \) in (iii), being adjacent either to all the vertices in \( V_i \) or to none. Accordingly, these \( r-1 \) vertices and \( v_{2r-2} \) induce either a \( K^r \) or a \( K^{2r} \) in \( G \), because \( v_i, \ldots, v_{2r-2} \in V_i \) for all \( i \).

\[\square\]

The least integer \( n \) associated with \( r \) as in Theorem 9.1.1 is the *Ramsey number* \( R(r) \) of \( r \); our proof shows that \( R(r) \leq 2^{2r-3} \). In Chapter 11 we shall use a simple probabilistic argument to show that \( R(r) \) is bounded below by \( 2^r/2 \) (Theorem 11.1.3).

In other words, the largest clique or independent set of vertices that a graph of order \( n \) must contain is, asymptotically, logarithmically small in \( n \). As soon as we forbid some fixed induced subgraph, however, it may be much bigger, of size linear in \( n \): The *Erdős-Hajnal conjecture* says that for every graph \( H \) there exists a constant \( \delta_H > 0 \) such that every graph \( G \) not containing an induced copy of \( H \) has a set of at least \( |G|^{\delta_H} \) vertices that are either independent or span a complete subgraph in \( G \).

It is customary in Ramsey theory to think of partitions as colourings: a *colouring* of (the elements of) a set \( X \) with \( c \) colours, or \( c \)-colouring for short, is simply a partition of \( X \) into \( c \) classes (indexed by the ‘colours’). In particular, these colourings need not satisfy any non-adjacency requirements as in Chapter 5. Given a \( c \)-colouring of \( |X|^k \), the set of all \( k \)-subsets of \( X \), we call a set \( Y \subseteq X \) *monochromatic* if all the elements of \( |Y|^k \) have the same colour,\(^1\) i.e. belong to the same of the \( c \) partition classes of \( |X|^k \). Similarly, if \( G = (V,E) \) is a graph and all the edges of \( H \subseteq G \) have the same colour in some colouring of \( E \), we call \( H \) a monochromatic subgraph of \( G \), speak of a red (green, etc.) \( H \) in \( G \), and so on.

In the above terminology, Ramsey’s theorem can be expressed as follows: for every \( r \) there exists an \( n \) such that, given any \( n \)-set \( X \),

\(^1\) Note that \( Y \) is called monochromatic, but it is the elements of \( |Y|^k \), not of \( Y \), that are (equally) coloured.
Indeed, as $X$ and $Y$ do not touch, the set $N(X)$ is disjoint from both $X$ and $Y$ and separates them in $G$. Hence $G$ has a separation $\{A, B\}$ such that $X \subseteq A \setminus B$ and $Y \subseteq B \setminus A$. As $|N(X)| \leq k$ since $X$ is a petal, choosing $\{A, B\}$ of minimum order ensures that $S := A \cap B$ has size at most $k$. By the minimality of $S$ and Menger’s Theorem 3.3.1, there is a family $\{P_s \mid s \in S\}$ of disjoint $S$-$N(X)$ paths in $G[A]$ and a family $\{Q_s \mid s \in S\}$ of disjoint $S$-$N(Y)$ paths in $G[B]$.

Let $H$ be the minor of $G$ obtained by deleting $A \setminus \bigcup_{s \in S} V(P_s)$ and contracting each of the paths $P_s$. Identifying the contracted branch sets $V(P_s)$ with their representatives $s$, we may think of $H$ as obtained from $G[B]$ by adding some edges on $S$. Let $(T_1, V_1)$ be the tree-decomposition which $(T_1, V_1)$ induces on $H$ as in Lemmas 12.3.2 and 12.3.3, and think of it as a tree-decomposition of $G[B]$. Thus for any $t \in T_1$, with the part $V_1^t \in V_1$ say, its part $V_t$ in $V_1^t$ is

$$V_t = (V_1^t \cap B) \cup \{s \in S \mid V_1^t \cap V(P_s) \neq \emptyset\}$$ (1)

(Fig. 12.4.1). In particular, $V_{x} = S$, since $V_x = X \cup N(X) \subseteq A$ and $N(X)$ meets every $P_s$. Similarly, let $J$ be the minor of $G$ obtained by deleting $B \setminus \bigcup_{s \in S} V(Q_s)$ and contracting the paths $Q_s$, and let $(T_2, V_2)$ be the tree-decomposition which $(T_2, V_2)$ induces on $J$. As before, think of this as a tree-decomposition of $G[A]$ in which $S$ is the part corresponding to $y$.

Let $T$ be obtained from the (disjoint) trees $T_1$ and $T_2$ by identifying $x$ and $y$ into a new node $r$. As $Y$ and $X$ are non-empty, $x$ is not the only node of $T_1$ and $y$ is not the only node of $T_2$, so $r$ is not a leaf of $T$. Let $V_r := S$. For all $t \in T - r$ let $V_t$ be the part in $V_1^t$ or $V_2^t$ that corresponds to $t$ there, thought of as a subset of $B$ if $t \in T_1$, or of $A$ if $t \in T_2$. We claim that $(T, V)$ with $V = (V_t)_{t \in T}$ is a good tree-decomposition of $G$ satisfying (1).

Using that $(T_1, V_1^t)$ and $(T_2, V_2^t)$ are tree-decompositions of $G[B]$ and $G[A]$, it is easy to check that $(T, V)$ is indeed a tree-decomposition of $G$. The non-leaves of $T$ are precisely those of $T_1$ and $T_2$, plus $r$. We have already seen that $|S| \leq k$. For $t \in T_1 - x$, its part $V_t$ in $V$ is no